InterlacedSperes.doc; 11.05.00

Interlaced Spheres and Multidimensional Tunnels

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Abstract

The paper presents some theorems about interlaced spheres of different dimensions in multidimensional spaces. Two spheres S^k and S^m are called interlaced with each other if their intersection is empty, however, one of them crosses each topological ball whose boundary is the other sphere. We describe a method of simulating interlaced spheres in computer. We demonstrate the connection between the notion of a tunnel in a multidimensional "body", i.e. in a connected subset of a multidimensional space, and that of interlaced spheres. Examples of multidimensional tunnels in four- and five-dimensional spaces are demonstrated.

1 Introduction

It is well-known that two closed curves in a three-dimensional Euclidean space can be interlaced with each other. This means that the curves do not intersect each other, however, each curve intersects any topological disk spanning the other curve. When considering a line segment as a one-dimensional disk D^1 and its boundary (which is a pair of points) as a zero-dimensional sphere S^0 (consisting of two end points of D^1) some obvious generalizations become possible: a pair of points may be interlaced with a closed curve in a 2D Euclidean space. This fact is another interpretation of the famous Jordan curve theorem. In the language of interlaced spheres it means that in a 2D Euclidean space an S^0 may be interlaced with an S^1. In Section 2 we prove a more general assertion: in an *n*-dimensional space homeomorphic to an *n*-dimensional ball an S^k may be interlaced with an S^m if and only if $m=n-k-1$ or $k+m+1=n$.

In section 3 we show how the Jordan theorem may be generalized for multidimensional spaces and spaces with disconnected boundaries.

It is easy to see that the common notion of a tunnel in a 3D body may be interpreted in the language of interlacing as follows: the body is interlaced with an S^1 which is not crossing the body, i.e. there exist in the body and on its surface a set *CC* of simple closed curves which do not intersect the said S^1, however each disk B^2 spanning a curve of *CC* intersects S^1 and vice versa: each disk spanning S^1 intersects all curves of *CC*.

We shall investigate in what follows the above supposition about the interlacing spheres, formulate the multidimensional version of the Jordan theorem, formulate the Jordan theorem for spaces which are subsets of two- and three-dimensional spheres and investigate tunnels in subsets of some simply connected multidimensional spaces. We also will demonstrate how to simulate all this situations in computer.

2 Basic notions

It is known [Moise 52] that manifolds of dimension 2 and 3 may be triangulated and that homeomorphic 2- and 3-manifolds are combinatorially homeomorphic. This knowledge may serve as the theoretical base for applying methods of combinatorial topology for the development of computer algorithms for topological investigations. Two complexes are called *combinatorially homeomorphic* if their simplicial schemata become isomorphic after finite sequences of elementary subdivisions [Still 95]. However, simplicial complexes contain too many elements and therefore are difficult to process. To overcome this drawback simplices may be united to greater cells by an operation inverse to the subdivision: a subcomplex combinatorially homeomorphic to a *k*-simplex (or equivalently to a *k*-ball) may be declared to be a *k*-dimensional cell or a *k*-cell. In what follows we shall write "homeomorphic" for "combinatorially homeomorphic".

While simplices are mostly considered as subsets of a Euclidean space we prefer to work with ACCs [Kov 89, Kov 92]. An ACC is a set of *abstract cells*. A non-negative integer is assigned to each cell. It is called the *dimension* of the cell. The set is provided with an antisymmetric, irreflexive and transitive binary relation called *bounding relation*. A cell can only bound another cell of higher dimension.

ACCs differ both from simplicial and Euclidean complexes in so far that an abstract cell is never a part of another cell. This property makes it possible to easily introduce the notion of *open subsets* of an ACC and thus to define a T_0-topology on it in accordance with classical axioms [Kov 89]. An ACC is not a quotient space of a Hausdorff space and thus it is independent form any such space. This is another advantage of the ACCs: a topological space which is a finite ACC may be *directly represented in computer*. Thus there is no necessity to consider theoretical problems in a Hausdorff space (which is not representable in computer) and then to transfer the results to a different set represented in the computer. All topological properties of ACCs and their elements may be directly computed in computer. This advantage of the ACCs is widely used in the present investigation. We represent topological spaces in computer as ACCs.

2 Interlaced Multidimensional Spheres

Consider an n–dimensional ACC A^n which is homeomorphic to an n-dimensional ball. In what follows we shall call such an ACC simply an n-ball and denote by B^n. The boundary ∂B^k of B^k is usually called a *standard* (k–1)-*sphere* and denoted by $S^{(k-1)}$. The ball B^k is said to span $S^k=\partial B^k$. Since we shall consider only standard spheres we shall call them simply *spheres*.

Definition IS: Two spheres S^m and S^k are called *interlaced* with each other if they do not intersect each other but each of them intersects every ball spanning the other sphere.

Definition BP: A subspace BP^m of an n-dimensional Cartesian ACC A^n [Kov 92], whose cells $c\in BP^m$ have m variable topological coordinates while the remaining n–m coordinates are constant, is called an *m-dimensional basic plane* of A^n.

Theorem IS: Two spheres S^m and S^k embedded in an n-dimensional ball B^n may be interlaced with each other if and only if $m+k+1=n$.

Proof: Let us show that such spheres really exist. Consider first the simpler case in which the ball B^n is an n-dimensional Cartesian ACC A^n. Consider an (m+1)-dimensional ball $B^{(m+1)}$ being a subset of an (m+1)-dimensional basic plane $E^{(m+1)}$ of A^n and a (k+1)-dimensional ball $B^{(k+1)}$ being a subset of a (k+1)-dimensional plane $E^{(k+1)}$. Let $S^m=\partial B^{(m+1)}$ and $S^k=\partial B^{(k+1)}$.

Consider now the intersection I of $B^{(k+1)}$ with $E^{(m+1)}$. It is obviously a ball of dimension not exceeding the minimum of k+1 and m+1. We shall show that I must be a one-dimensional ball, i.e. a line segment. Really, if I has a dimension greater than one then its boundary ∂I is a sphere of dimension greater than zero and therefore ∂I is connected. There are two possibilities:

a) I lies completely in $B^{(m+1)}$ without crossing its boundary. Then, there is also no crossing of $B^{(k+1)}$ with S^m, which contradicts the Definition IS. Really:

$$I\cap\partial B^{(m+1)}=\varnothing \rightarrow (B^{(k+1)}\cap E^{(m+1)})\cap\partial B^{(m+1)}=\varnothing \rightarrow$$

$$B^{(k+1)}\cap(E^{(m+1)}\cap\partial B^{(m+1)})=\varnothing \rightarrow B^{(k+1)}\cap\partial B^{(m+1)}=\varnothing\rightarrow B^{(k+1)}\cap S^m=\varnothing;$$

b) I crosses $\partial B^{(m+1)}$. Then there are two points in I: one in the interior and one in the exterior of $B^{(m+1)}$. If the dimension of I is greater than one, then ∂I is a sphere of dimension greater than 0 and is connected. Then ∂I must cross $\partial B^{(m+1)}=S^m$. Since ∂I is a subset of S^k it follows that S^k crosses S^m, which contradicts the Definition IS. If however the dimension of I is one then ∂I is a sphere of dimension 0. It consists of two disjoint points and thus is disconnected. Therefore S^k does not cross S^m.

The dimension of I cannot be 0 since in that case I would consist of a single point. However, according to Definition IS $I=B^{(k+1)}\cap E^{(m+1)}$ contains the intersection $S^k\cap B^{(m+1)}$ since $S^k\subset B^{(k+1)}$ and $B^{(m+1)}\subset E^{(m+1)}$. It also contains the intersection $S^m\cap B^{(k+1)}$ since $S^m\subset E^{(k+1)}$. But the first intersection lies in the interior of $B^{(m+1)}$ (S^k and $\partial B^{(m+1)}$ do not intersect), while the second intersection lies in $\partial B^{(m+1)}=S^m$. One point cannot lie both in the interior and in the boundary.

Thus we have proved that the dimension of the intersection of $B^{(k+1)}$ with $E^{(m+1)}$ is exactly one. Since these sets are subspaces of the n-dimensional space A^n the sum of their dimensions minus 1 (the dimension of their intersection) cannot exceed n: $k+1+m+1-1 \le n$ or $k+m+1 \le n$. It remains to show that $k+m+1=n$.

According to the condition, S^k must cross any ball spanning S^m. According to the above consideration the ball $B^{(m+1)}$ lies in the same $(m+1)$-dimensional plane $E^{(m+1)}$ as S^m itself. The ball $B^{(m+1)}$ is crossed by S^k. However if the dimension n of the space is greater than $k+m+1$ then there is at least one degree of freedom outside of the two planes $E^{(k+1)}$ and $E^{(m+1)}$. This degree of freedom may be used to construct another ball B' of dimension $m+1$ having the same boundary S^m and not crossing S^k. Really:

Let $\boldsymbol{v}$ be the coordinate vector orthogonal to both $E^{(k+1)}$ and $E^{(m+1)}$. Then the ball $B^{(m+1)}$ being translated by $\boldsymbol{v}$ and united with the direct product $\boldsymbol{v} \times S^m$ composes the ball

$$B' = T(B^{(m+1)}, \boldsymbol{v}) \cup \boldsymbol{v} \times S^m;$$

where $T(B^{(m+1)}, \boldsymbol{v})$ denotes the ball $B^{(m+1)}$ translated by $\boldsymbol{v}$.

The boundary of B' is again S^m since the boundary of $\boldsymbol{v} \times S^m$ consists of two spheres one of which is S^m and the other is identical with the boundary of $T(B^{(m+1)}, \boldsymbol{v})$ and thus does not belong to the boundary of B'. The first term of B' lies in the plane $T(E^{(m+1)}, \boldsymbol{v})$ not intersecting $E^{(k+1)}$. The second term is orthogonal to $E^{(k+1)}$. Hence neither of them intersects S^k.

Thus we have shown, that there exist in an n-dimensional ACC A^n two interlaced spheres of dimensions k and m if and only if $k+m+1=n$. The spheres lie in two planes of dimensions $k+1$ and $m+1$ respectively while the intersection of the planes is one-dimensional. It may be shown that any two standard spheres of the same dimensions may be interlaced with each other if there exists such a homeomorphism of the space A^n onto itself which brings the spheres into the corresponding planes.

Supposition SP. For any k-dimensional standard sphere S^k in an n-dimensional space A^n there exists a homeomorphism of A^n onto itself which puts S^k into a $(k+1)$-dimensional basic plane.

Corollary IS: For any k-dimensional sphere S^k in an n-dimensional space satisfying the above supposition there exists an $(n-k-1)$-dimensional sphere interlaced with S^k.

Proof: After having transformed the space (according to Supposition SP) in such a way that S^k lies in a $(k+1)$-dimensional plane $E^{(k+1)}$ one can construct another, $(m+1)$-dimensional plane $E^{(m+1)}$ with $m=n-k-1$ whose intersection with $E^{(k+1)}$ is one-dimensional. The plane $E^{(m+1)}$ must intersect S^k. Let $P \in E^{(m+1)} \cap S^k$ be a point in the intersection and $B^{(m+1)} \subset E^{(m+1)}$ any $(m+1)$-dimensional ball containing P. The boundary $\partial B^{(m+1)}$ satisfies the conditions of Theorem IS and thus it is the desired interlaced sphere.

3 Examples of interlaced spheres

Fig. 1 represents some examples of interlaced spheres in spaces of dimension from 0 to 3. An important special case is that of $m=0$ and $k=n-1$. In this case one of the spheres consists of two disjoint points. According to Theorem IS any simple curve connecting the points crosses the other sphere $S^{(n-1)}$ of dimension $n-1$. We shall show in Section 4 (Theorem MJ) that for all $n>1$ the rest of the space is subdivided by $S^{(n-1)}$ into exactly two components which is the contents of the multidimensional Jordan theorem for spheres.

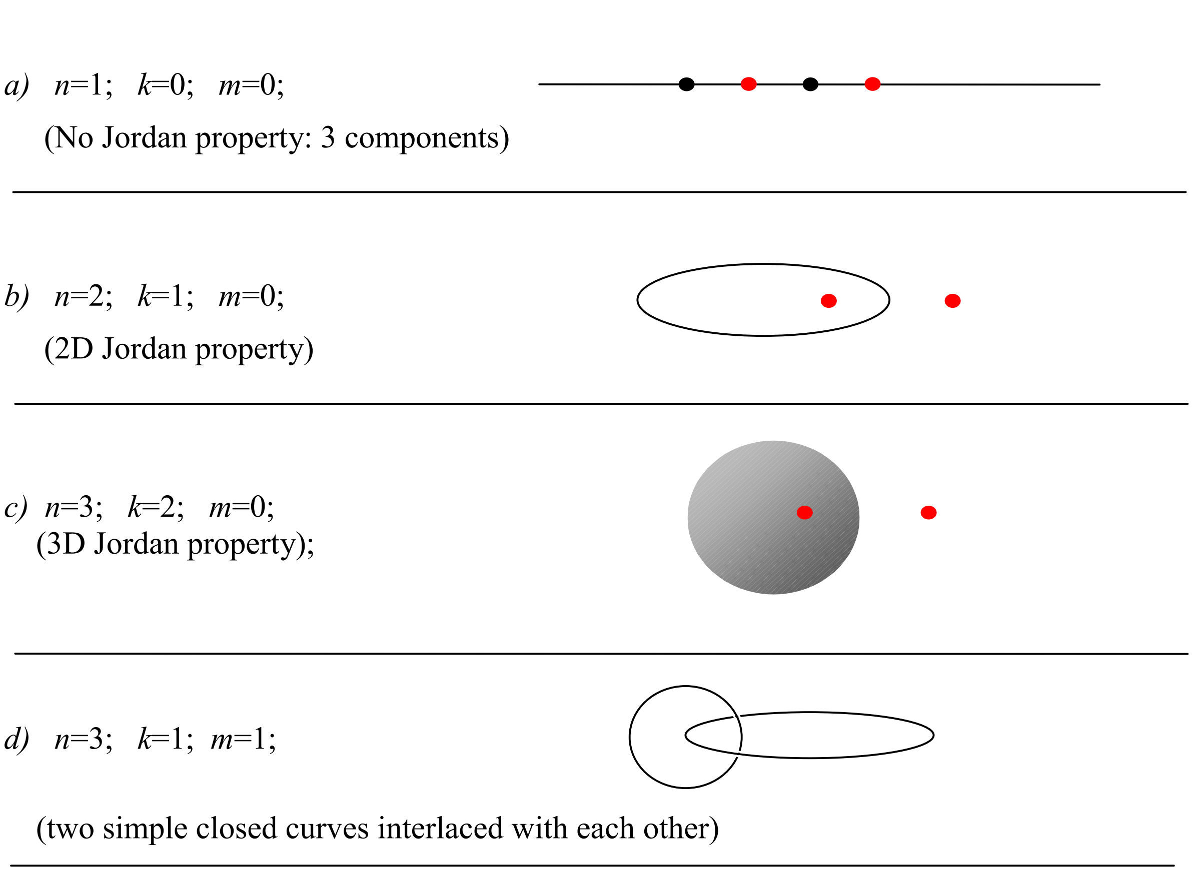

Fig. 1. Examples of interlaced spheres in spaces of dimension 1 to 3

Examples for higher dimensions may be produced by means of computer. We have represented an n-dimensional Cartesian ACC A^n with n=4 and 5 as an n-dimensional array of bytes. The indices x, y, z, t and u (the latter only in the 5D case) of the array are the topological coordinates. Thus the array contains a byte for each cell of A^n of any dimension from 0 to n.

We have labeled the bits of the bytes according to the membership of the corresponding cell in different subsets as follows (Fig. 2):

bit 0 (label 1) for cells of Int($B^{(k+1)}$, $E^{(k+1)}$);

bit 1 (label 2) for cells of $\partial B^{(k+1)}$;

bit 2 (label 4) for cells of Int($B^{(m+1)}$, $E^{(m+1)}$);

bit 3 (label 8) for cells of $\partial B^{(m+1)}$;

Here Int($B^{(k+1)}$, $E^{(k+1)}$) denotes the interior of $B^{(k+1)}$ relative to the subspace $E^{(k+1)}$ which means that Int($B^{(k+1)}$, $E^{(k+1)}$) does not contain $\partial B^{(k+1)}$. The same is true for Int($B^{(m+1)}$, $E^{(m+1)}$).

If a cell is in the intersection of some of the above mentioned subsets then its byte has all the bits corresponding to the subsets. For example, in the intersection Int($B^{(k+1)}$)$\cap\partial B^{(m+1)}$ both bits 0 and 3 must be set. Thus the corresponding byte must have the label 1+8=9 etc.

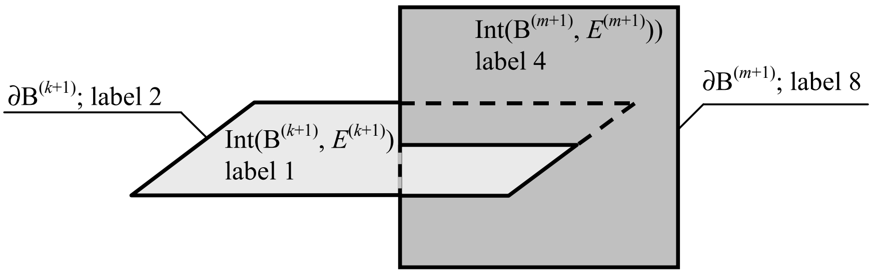

Fig. 2. Assigning labels to balls and spheres

The program has labeled as Int($B^{(k+1)}$) an three-dimensional parallelepiped (k=2) in the plane t(=u)=Const and as Int($B^{(m+1)}$) an (n–k)-dimensional parallelepiped in the orthogonal plane x=y=Const. Then the program has labeled the boundaries of the parallelepipeds as $\partial B^{(k+1)}$ and $\partial B^{(m+1)}$ respectively and computed a histogram of labels. The histograms for the cases of 4D and 5D are represented in Table 1 and Table 2 respectively.

Table 1
Number of cells of different dimensions and labels in the 4D ACC; k=2; m=1.

	Number of cells with label:								
Dim	0	1	2	4	5	6	8	9	10
0	624	6	55	1	1	**1**	11	**1**	**0**
1	2636	34	108	8	2	0	12	0	**0**
2	4084	54	54	8	0	0	0	0	**0**
3	2773	27	0	0	0	0	0	0	**0**
4	700	0	0	0	0	0	0	0	**0**

As one can see, the intersection of the spheres, which should have the label 2+8=10 is empty while the intersection of S^k with $B^{(m+1)}$ (label 6) as well as the intersection of S^m with $B^{(k+1)}$ (label 9) contain one point (0-cell) each.

Table 2
Number of cells of different dimensions and labels in the 5D ACC; $k=m=2$.

	Number of cells with label:								
Dim	0	1	2	4	5	6	8	9	10
0	2694	6	55	1	1	**1**	41	**1**	**0**
1	13762	34	108	14	2	0	80	0	**0**
2	27824	54	54	28	0	0	40	0	**0**
3	27957	27	0	16	0	0	0	0	**0**
4	14000								
5	2800	0	0	0	0	0	0	0	**0**

The intersections are again in full correspondence with the theory.

The list of possible interlaced spheres in spaces of different dimensions, which was started with Fig. 1, may be continued as follows:

e) $n=4$; $k=3$; $m=0$; (4D Jordan property);

f) $n=4$; $k=2$; $m=1$; (S^2 interlaced with S^1, as represented in Table 1);

g) $n=5$; $k=4$; $m=0$; (5D Jordan property);

h) $n=5$; $k=3$; $m=1$; (S^3 interlaced with S^1);

i) $n=5$; $k=2$; $m=2$; (S^2 interlaced with S^2, as represented in Table 2).

4 The Jordan property in multidimensional spaces

The validity of the famous Jordan theorem in finite or "discrete" spaces was multiply discussed in the literature. The property of a subset $SS\subset S$ of the space S to decompose its complement $S-SS$ into exactly two components is often called the *Jordan property* of SS. It was proven in [Brower 12] that an $(n-1)$-dimensional sphere embedded into the n-dimensional Euclidean space possesses the Jordan property. However, this is not the most general case: the Jordan property depends on both: the nature of the space and on that of the subset. From the point of view of applications it is important to know which subsets of which spaces possess the Jordan property. We shall investigate in what follows the Jordan property of some subsets of ACCs.

First of all it should be noticed that the frontier of a subset often has the Jordan property. However, the conditions for this must be investigated. One may find in a text book for topology (e.g. in [Rin 75] that if SC is a subset of a space S then

$$S=\text{Int}(SC, S)\cup\text{Fr}(SC, S)\cup\text{Ext}(SC, S) \qquad (1)$$

while each two of the three sets on the right hand side of (1) are disjoint. It is of course not enough for the Jordan property since some of the sets may be empty or disconnected. To specify the conditions for the Jordan property we need the notion of a simple frontier:

Definition SF: The frontier F of an n-dimensional subcomplex SC of an n-dimensional ACC A^n is called *simple* if for each cell $c\in F$ the intersection of the SON*(c) with both SC and its complement A^n-SC is not empty and connected.

Fig. 3 shows some examples of subcomplexes whose frontiers are not simple.

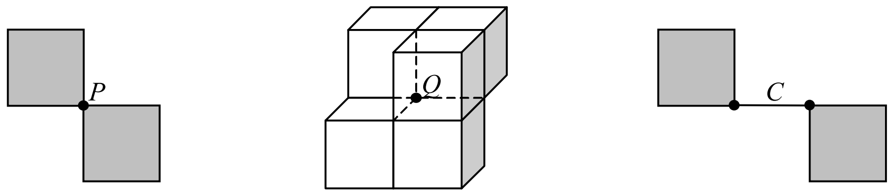

Fig. 3. Examples of subcomplexes whose frontiers are not simple

In Fig. 3a both intersections of SON*(P) with the shaded two-dimensional complex and with its complement (the background) are disconnected. In Fig. 3b the intersections of SON*(Q) with the three-dimensional complex containing six voxels (one of them is invisible) is connected, but the intersection with the background (two voxels) is disconnected. In Fig. 3c the intersection of SON*(C) of a 1-cell C with the shaded two-dimensional complex containing C is empty since C does not belong to SON*(C).

Consider a finite topological space which is an n-dimensional ACC A^n homeomorphic to a closed n-ball. The boundary ∂A^n is then homeomorphic to a sphere. Let us call the union of the SONs of all cells of ∂A^n the *boundary shell* of A^n.

Definition BS: An n-dimensional subcomplex SC of A^n is called *bounded in* A^n if it does not cross the boundary shell of the space.

Theorem MJ: The frontier Fr(SC) of an n-dimensional bounded subcomplex SC of an n-dimensional Cartesian ACC A^n possesses the Jordan property if and only if it is connected and simple.

Proof: Consider a cell c_1 of SC and choose an arbitrary coordinate axis X of A^n. Consider a sequence of mutually incident cells (c_1, c_2, ..., c_m) such that for all cells of the sequence all topological coordinates but the X-coordinate are equal to that of c_1 while the X-coordinate increases from one cell to the subsequent one by 1:

$$X(c_{(i+1)})=X(c_i)+1.$$

The first cell of the sequence is the given cell c_1, the last one c_m is a cell in the boundary shall of the space and thus does not belong to SC. Somewhere in the sequence there must be a cell f_1 belonging to the frontier F(SC). Let us call it the *frontier projection* of c_1. In the same way one may find the frontier projection f_2 of any other cell c_2 of SC. Since the frontier F(SC) is connected there exists a path PF in F(SC) connecting f_1 with f_2. We shall demonstrate that if F(SC) is simple then there exists another path lying in the interior SC–F(SC) and connecting c_1 with c_2.

According to Definition SF, each cell of a simple frontier contains in its SON* a connected subset of SC. Consider two incident cells c' and c'' of the path PF in the frontier F(SC). One of them, say c', is bounding another, say c''. Then SON(c'')$\subset$SON(c') and

$$\text{SON*}(c'')\cap SC \subset \text{SON*}(c')\cap SC .$$

Therefore the union SON*(c_i)$\cap SC\cup$SON*(c_{i+1})$\cap SC$ for two subsequent cells c_i and c_{i+1} in the path PF is connected. So is the union of the subsets SON*(c_i)$\cap SC$ of all cells c_i of PF. All these subsets lie in SC–F(SC) since the cells of F(SC) are not contained in SON*(c_i). The union of PF with the paths from c_1 to f_1 and from c_2 to f_2 composes a path in SC–F(SC)

connecting c_1 with c_2. This may be shown for any two cells of SC–F(SC). Therefore the set SC–F(SC) is connected. It may be shown a similar way that the set A^n–SC–F(SC) is also connected.

It would be interesting to prove the following supposition:

Supposition JM: A connected oriented and bounded (n–1)-dimensional manifold $M^{(n-1)}$ without boundary, being embedded into an n-dimensional space SP^n homeomorphic to an n-dimensional ball B^n, decomposes SP^n into two components.

To prove the supposition it would be enough to show that any such manifold is a simple boundary of some connected n-dimensional subset of B^n, however, it is not easy to show this.

Theorem FN: Let A^n be a connected n-dimensional ACC, G^k and H^k two connected k-dimensional with $k \le n$ subsets of A^n such that $G^k \subset H^k$ and the complement $H^k - G^k$ is also k-dimensional. Then the frontier Fr(G^k, H^k) decomposes H^k, i.e. H^k–Fr(G^k, H^k) is disconnected.

Proof: Consider H^k as a topological space. Then H^k consists of three disjoint subsets:

$$H^k = \mathrm{Int}(G^k, H^k) \cup \mathrm{Fr}(G^k, H^k) \cup \mathrm{Ext}(G^k, H^k); \qquad (3)$$

i.e. the intersection of each two subsets of the right hand part of (3) is empty. Thus

$$H^k - \mathrm{Fr}(G^k, H^k) = \mathrm{Int}(G^k, H^k) \cup \mathrm{Ext}(G^k, H^k); \qquad (4)$$

The sets in the right hand part of (4) are both open in H^k and disjoint. Therefore there is no space element incident with both Int(G^k, H^k) and Ext(G^k, H^k). It follows that there exits no incidence path between Int(G^k, H^k) and Ext(G^k, H^k).■

Lemma NB: Consider two n-manifolds M_1 and M_2 with boundaries while $M_1 \subset M_2$ is bounded in M_2. Let P be a point of ∂M_1. Consider two neighborhoods of P: N_2=SON(P, M_2) is the neighborhood of P in M_2 and N_1=SON(P, ∂M_1) is the neighborhood of P in ∂M_1. N_1 is obviously a subset of N_2. Then the set $N_2 - N_1$ consists of two components.

Proof: Since M_1 is n-dimensional and bounded in M_2 the boundary ∂M_1 coincides with the frontier Fr(M_1, M_2). Therefore N_1 is the frontier of $N_2 \cap M_1$ relative to N_2: N_1=Fr($N_2 \cap M_1$, N_2). According to Theorem FN $N_2 - N_1$ is disconnected and it is the union of two disjoint subsets: $N_2 \cap$Int(M_1, M_2) and $N_2 \cap$Ext(M_1, M_2). M_1 is a manifold with boundary and P lies in its boundary. Thus according to the definition of a manifold with boundary each of the subsets $N_2 \cap$Int(M_1, M_2) and $N_2 \cap$Ext(M_1, M_2) is the interior of a half-ball and hence is connected. Therefore the number of components of $N_2 - N_1$ is exactly two.■

Consider the special case of interlaced spheres S^k and S^m with m=0. According to the above Corollary for each $S^{(n-1)}$ in an n-dimensional space A^n there exists an S^0 interlaced with $S^{(n-1)}$. The sphere S^0 consists of two points, say P_1 and P_2. One of them, say P_1, lies in the ball spanned by $S^{(n-1)}$ (since the dimension of the sphere is n–1 there exists only one such ball), the other lies outside. According to the definition IS any simple curve connecting P_1 with P_2 crosses $S^{(n-1)}$. Therefore P_1 and P_2 are disconnected in $A^n - S^{(n-1)}$ and the set $A^n - S^{(n-1)}$ is decomposed by $S^{(n-1)}$ into exactly two components: the set of all cells of A^n connected with P_1 and that of all cells connected with P_2. This is what we call the Jordan property of $S^{(n-1)}$.

Corollary JS: A k-dimensional sphere S^k in an n-dimensional space A^n homeomorphic to an n-dimensional ball decomposes the space A^n into two components if and only if k=n–1.

The Jordan property of spheres may be generalized in the following two ways:

a) for manifolds other than spheres;

b) for spaces which are balls with disconnected boundaries.

5 Multidimensional tunnels

Interlaced spheres in an n-dimensional space may be "thickened" so that each 0-cell of a sphere be united with its SON. In this way an m-dimensional sphere S^m, $m<n$, becomes transformed to an open n-dimensional subset V. It may be thickened once more by applying the above operation onto the 0-cells of the closure $Cl(V)$ etc. The sphere S^m obviously lies in the interior of V. If the number of the steps of thickening is not too great in comparison with the size of the sphere S^k, $k=n-m-1$, interlaced with S^m before the thickening, then the spheres remain interlaced also after the thickening.

For example, if one applies the thickening procedure onto one of two interlaced one-dimensional spheres in a three-dimensional space one obtains a solid torus while the other sphere is interlaced with each parallel of the torus. (We denote as a *parallel* each one-dimensional sphere on the torus surface, which possess a spanning disk not intersecting the interior of the torus; spheres whose all spanning disks intersect the interior are called *meridians*).

If a connected n-dimensional subset V of an n-dimensional space contains a sphere interlaced with another k-dimensional sphere S^k not intersecting V then we say that V has a *tunnel of dimension k*. Thus a torus in a three-dimensional space has a one-dimensional tunnel.

Let us call the k-dimensional sphere S^k not intersecting V and interlaced with some sphere S^m lying in ∂V, the *axis* of the k-dimensional tunnel of V and the sphere S^m the *equator* of the tunnel of V.

The axis of a tunnel may also be thickened by the procedure described above. If the number of the steps of thickening is not too great one obtains two n-dimensional subsets V_1 and V_2 each of which contains one of two interlaced spheres. We shall call such subsets interlaced with each other. In the case of $n=3$ two links of a chain (in the common sense, rather than in the sense of algebraic topology) may serve as an example of interlaced three-dimensional subsets.

A simple example of a two-dimensional tunnel in a four-dimensional ACC A^4 may be constructed as follows. As we know, there exist in A^4 an S^2 interlaced with an S^1. Let us take an example of S^1 in A^4 and thicken it to a four-dimensional subset V^4. Then we can found a concrete S^2 which is a two-dimensional axis of the two-dimensional tunnel in V^4.

We shall represent the example as a list of the coordinate quadruples. To make it as simple as possible we take the number of cells small. The subset *V* consists of 12 four-dimensional cells. Their topological coordinates we denote by *x*, *y*, *z*, *t*. All these cells have constant values of the coordinates *z* and *t* and thus may be well represented by their projections onto the two-dimensional coordinate plane XY:

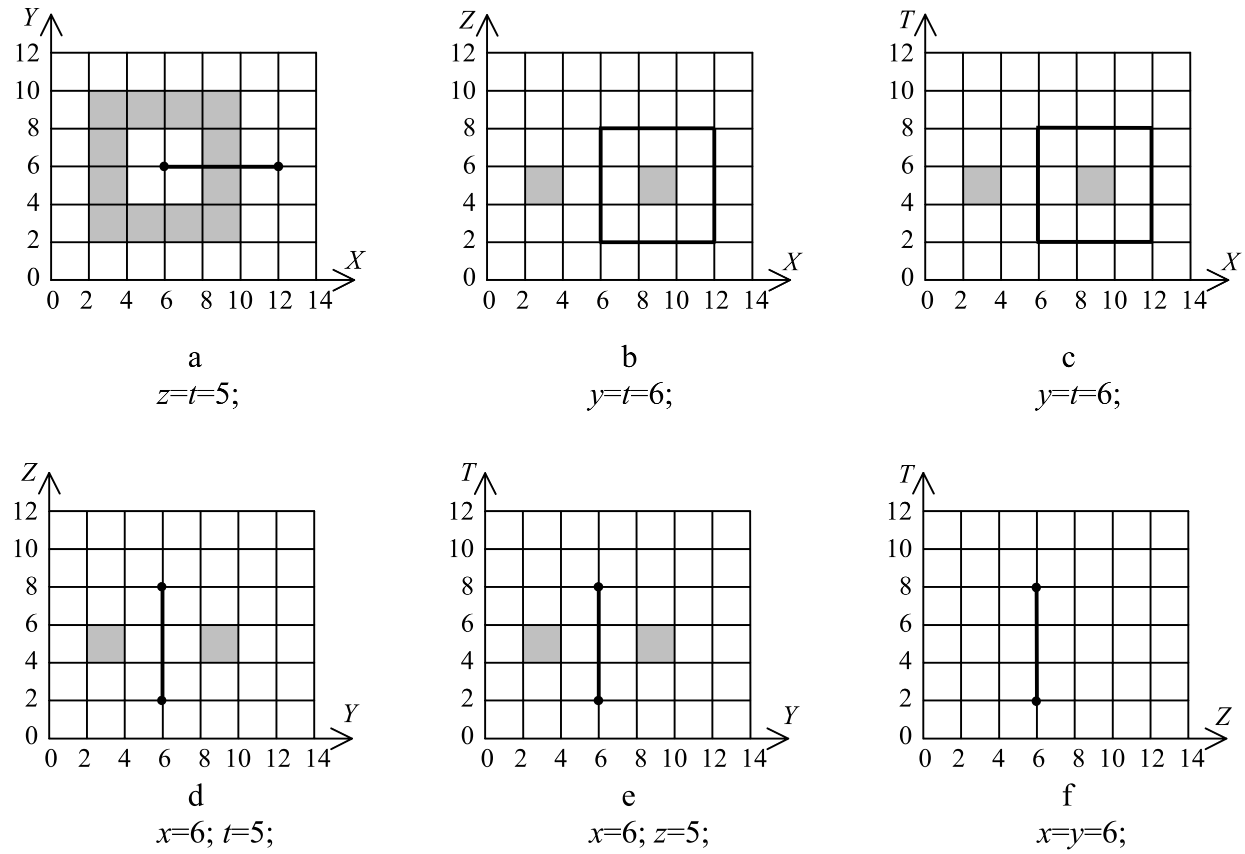

Fig. 4. Six 2D cross sections through the 4D ACC containing V^4 and S^2

The 4-cells of V^4= {(3,9,5,5),(5,9,5,5),(7,9,5,5), (9,9,5,5),
(3,7,5,5), (9,7,5,5), (1)
(3,5,5,5), (9,5,5,5),
(3,3,5,5),(5,3,5,5),(7,3,5,5), (9,3,5,5)};

The subset V^4 consists of the 4-cells of (1) and all cells of the interior of the union of the closures of these 4-cells. (Thus we have defined an open set, which is only important to have as few as possible cells). In this case V^4 contains each 3-cells lying between two adjacent 4-cells, e.g. the 3-cell (4,9,5,5) lies between the 4-cells (3,9,5,5) and (5,9,5,5). The axis S^2 consists of all 2-cells of the 3D cube D^3 defined by the inequalities:

$$6 \le x \le 12;$$
$$y = 6; \quad (2)$$
$$2 \le z \le 8;$$
$$2 \le t \le 8;$$

and all cells of dimension 0 and 1, belonging to their closures. These cells lie in six 2D planes containing the boundary of the cube. The plane with the equation *x*=6; y=6; (*z* and *t* may take arbitrary values) does not cross V^4: none of the cells of V^4 has *x*=6 and y=6. However, the plane crosses the convex hull of V^4, which contains e.g. the 2-cell C^2=(6,6,5,5); Really, the coordinates of C^2 are convex combinations of the coordinates of the cells (3,7,5,5) and (9,5,5,5). (This means 6=(3·(9–6)+9·(6–3))/(9–3)). Thus the two-dimensional plane *x*=6; y=6; is a two-dimensional axis (different from S^2) of the tunnel in V^4.

Sets possessing tunnels of dimensions greater then 2 may be constructed in spaces of dimensions greater then 4 in a similar way.

Proposition MT: A tunnel of dimension *m* may exist in a space of dimension at least *m*+2.

Proof: The set V^n possessing a tunnel must be a thickened sphere S^k (or a connected sum of such spheres) of dimension $k \geq 1$ since V^n must be connected and S^0 is disconnected. S^k may be interlaceded with S^m only in a space of dimension $n=k+m+1$. From $k \geq 1$ follows $n \geq m+2$.■

Some time Yung Kong said to the author: "It is easy to answer the question, *where* is a cavity in a body, but it is impossible to say, where is the tunnel". We shall try to answer the question:

> The tunnel of *V* lies in the component *C* of the set Conv(*V*)–*V*, whose common boundary with *V* (i.e. the set $\partial C \cap \partial V$) contains spheres interlaced with the axis of the tunnel.

Here Conv(*V*) denotes the digital convex hull of *V*.

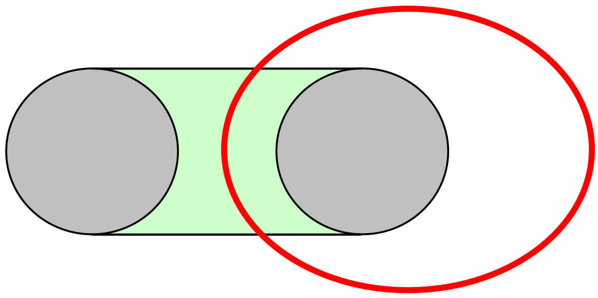

Fig. 5. Location of the tunnel in a torus

The notion of the digital convex hull may be deduced in the obvious way from our earlier definition of a digital convex subset of an Cartesian ACC, which was defined for a two-dimensional ACC [Kov 92]. Let us formulate the notions for the *n*-dimensional case.

Definition HN: A complex *C* (especially an ACC) is called *homogeneously n-dimensional* if each cell of *C* is incident with an *n*-cell of *C*.

Definition SD: A homogeneously *n*-dimensional subcomplex *SC* of an *n*-dimensional ACC A^n is called *solid* if its complement A^n–*SC* is also homogeneously *n*-dimensional.

Definition HS: A solid n-dimensional subcomplex *SC* of an n-dimensional Cartesian ACC A^n, whose cells have topological coordinates satisfying a linear inequality is called a *digital half-space* of A^n.

It is of no importance, whether the coefficients of the inequality are rational numbers or real ones since we consider only finite ACCs. For any inequality with real coefficients there exits an inequality with rational coefficients defining the same subset of a finite ACC.

Definition CV: An non-empty intersection of a finite number of digital half-spaces is called a *digital convex set*.

Definition CH: The digital convex set containing a given set V and possessing the minimum number of cells is called the convex hull of V.

A three-dimensional volume V whose surface S is a 2-manifold of genus G has G tunnels. It is not so easy, as in the case of a single tunnel, to specify their location. One possible way of doing so consists in the following: the fundamental group of S has $2 \cdot G$ generators. Among the curves (one-dimensional spheres) corresponding to the generators there are G curves interlaced with curves inside V and G other curves interlaced with curves not intersection V. The letter are the axes of the tunnels; they are interlaced with the meridians of S. Now the location of the tunnels may be defined as specified above. In spaces of dimension greater then 3 higher homotopy groups must be considered.

6 Conclusion

Spheres in multi-dimensional spaces may exhibit properties which are quite unusual from the point of view of inhabitants of a three-dimensional space. These properties are on the one side connected to generalization of the famous Jordan theorem for multidimensional spaces. On the other side they enable us to generalize the notion of a tunnel for subsets of multidimensional spaces.

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